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Suppose that $a \geq 0, b \geq 0, c \geq 0, a+b+c=10.$ Find the maximum value of

$$(a-b)(b-c)(c-a).$$

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First we will find $\max\{(a-b)(b-c)(c-a) \mid a \geq 0, b \geq 0, c \geq 0, a+b+c=1\}.$

Let $P(a,b,c) := (a-b)(b-c)(c-a).$ Since $P(b,a,c) = (b-a)(a-c)(c-b) = -(a-b)(b-c)(c-a) = -P(a,b,c)$ then

$$\max\{P(a,b,c) \mid a \geq 0, b \geq 0, c \geq 0, a+b+c=1\} = \max\{|P(a,b,c)| \mid a \geq 0, b \geq 0, c \geq 0, a+b+c=1\}$$

Noting that $|P(a,b,c)|$ is symmetric we can assume that $a \geq b \geq c.$

Then denoting $x := b-c, y := a-b$ we obtain $b = c+x, a = c+x+y, a+b+c=1 \Leftrightarrow 3c+2x+y=1$ and $|P(a,b,c)| = (a-b)(b-c)(a-c) = yx(x+y),$ where $2x+y \leq 1$ and $c = \dfrac{1-2x-y}{3}.$ For any $p,q > 0$ by AM-GM inequality we have

$$xy(x+y) \leq x(1-2x)(1-x) = \frac{1}{pq}(p-2px)x(q-qx) \leq \frac{2}{pq}\left(\frac{p-2px+x+q-qx}{3}\right)^3 = \frac{2}{pq}\left(\frac{p+q+x(1-q-2p)}{3}\right)^3.$$

We claim $2p+q=1, p-2px=x \Leftrightarrow x = \dfrac{p}{2p+1}, x = \dfrac{q}{q+1}$

Hence, $\begin{cases} 2p+q=1 \\ \dfrac{p}{2p+1} = \dfrac{q}{q+1} \end{cases} \Leftrightarrow (p,q) = \left(\dfrac{\sqrt{3}-1}{2}, 2-\sqrt{3}\right).$

Thus, $xy(x+y) \leq \dfrac{1}{\dfrac{\sqrt{3}-1}{2}\cdot\left(2-\sqrt{3}\right)}\left(\dfrac{\dfrac{\sqrt{3}-1}{2}+2-\sqrt{3}}{3}\right)^3 = \dfrac{\sqrt{3}}{18}.$

Since equality occurs iff $x = \dfrac{\dfrac{\sqrt{3}-1}{2}}{2\cdot\dfrac{\sqrt{3}-1}{2}+1} = \dfrac{3-\sqrt{3}}{6},\ y = 1-2\cdot\dfrac{3-\sqrt{3}}{6} = \dfrac{\sqrt{3}}{3}$

then $\max\{(a-b)(b-c)(a-c) \mid a \geq b \geq c \geq 0, a+b+c=10\} = \dfrac{\sqrt{3}}{18}$

and attained if $c = 0, b = \dfrac{3-\sqrt{3}}{6}, a = \dfrac{3-\sqrt{3}}{6} + \dfrac{\sqrt{3}}{3} = \dfrac{3+\sqrt{3}}{6}.$

Therefore, $\max\{P(a,b,c) \mid a \geq 0, b \geq 0, c \geq 0, a+b+c=1\} = \dfrac{\sqrt{3}}{18}$

and attained if $(a,b,c) = \left(\dfrac{3-\sqrt{3}}{6}, \dfrac{3+\sqrt{3}}{6}, 0\right).$

Or, in the form of homogeneous inequality, namely:

For any real $a,b,c \geq 0$ holds inequality

(1) $(a-b)(b-c)(c-a) \leq \dfrac{\sqrt{3}}{18}(a+b+c)^3$ with equality if

$$(a,b,c) = k\left(\frac{3-\sqrt{3}}{6}, \frac{3+\sqrt{3}}{6}, 0\right), \forall k > 0.$$

Hence, if $a+b+c=10$ we obtain $(a-b)(b-c)(c-a) \leq \frac{\sqrt{3}}{18} \cdot 10^3 = \frac{500\sqrt{3}}{9}$

with equality if $(a,b,c) = 10\left(\frac{3-\sqrt{3}}{6}, \frac{3+\sqrt{3}}{6}, 0\right) = \left(\frac{5\left(3-\sqrt{3}\right)}{3}, \frac{5\left(3+\sqrt{3}\right)}{3}, 0\right)$

that is $\max\{P(a,b,c) \mid a \geq 0, b \geq 0, c \geq 0, a+b+c=10\} = \frac{500\sqrt{3}}{9}$.